

## BEYOND UNCOUNTABLE

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... The fact is that such a procedure is not applicable. Why? Because their definitions are not predicative and contain within such a vicious circle I already mentioned above; not predicative definitions can not be substituted to defined terms. In this condition, *logistics is no longer sterile: it generates contradictions*. (Jules-Henri Poincaré 1902, [10] 211, our translation.)

## INTRODUCTION

By common consent Russell's antinomy is the reason why in Zermelo–Fraenkel set theory, there is no set which comprehends *all* sets. Furthermore, given any set  $A$ , there is no set which contains *all* sets which are not members of  $A$  (in particular, there is no set which is the complement of  $A$ ) ([7] 40–41). In other words, given any set  $A$ , the *absolute* complement of  $A$ , i.e.  $\{x \mid x \notin A\}$ , cannot be defined and the complement of  $A$ , can only be defined as *relative* to another given set. For instance, if  $A$  is a subset of  $B$ , then the relative complement of  $A$  in  $B$  is defined by

$$B - A = \{x \in B \mid x \notin A\}.$$

The existence of the relative complement is ensured by the axiom schema of the Subsets

$$(1) \quad \forall z_1 \dots z_n \forall s \exists y \forall x (x \in y \iff x \in s \wedge \varphi(x)),$$

where  $\varphi(x)$  is a first order well formed formula,  $z_1, \dots, z_n, x$  are the free variables of  $\varphi(x)$ , and  $y$  is not free in  $\varphi(x)$ , which admits general comprehension only for members  $x$  of a given set  $s$ . Indeed we are always allowed to assert

$$(2) \quad \forall z \forall s \exists y \forall x (x \in y \iff x \in s \wedge x \notin z),$$

as an instance of (1). This set  $y$  is the relative complement of  $z$  in  $s$  ([6] 23). This premise and the following subsection are introductory to the results of Section 1. In this abstract Zermelo–Fraenkel set theory stands for general first order set theory.

*Basic setup.* We refer to Zermelo–Fraenkel set theory with  $\mathfrak{ZF}$ . Let us recall the axiom of Extensionality

$$(3) \quad \forall x \forall y [\forall z (z \in x \iff z \in y) \implies x = y],$$

and the other main concepts we shall be concerned with. For details see [6, 7].

$$x \subseteq z = \forall w (w \in x \implies w \in z),$$

$$P(z) = \{x \mid x \subseteq z\},$$

$x \sim y$  denotes that the sets  $x$  and  $y$  are equinumerous or equal in cardinality, namely there exists a one to one correspondence between their elements.

$x \not\sim y$  denotes that the sets  $x$  and  $y$  are not equal in cardinality, namely there exists no one to one correspondence between their elements.

$x <_c y$  and  $x \leq_c y$  denote respectively that the set  $x$  has cardinality properly less than the cardinality of  $y$ , and that the set  $x$  has cardinality less than  $y$ .

Let us also recall the argumentation of the so-called Cantor's theorem. We shall present the version in ([9] 15), for a more detailed exposition the reader is referred to ([1, 2],[6]).

(Cantor's proposition). For every set  $A$  in  $\mathfrak{F}$ ,

$$A <_c P(A)$$

i.e.  $A \leq_c P(A)$  but  $A \not\sim P(A)$ .

That  $A \leq_c P(A)$  follows from the fact that the function

$$x \mapsto \{x\}$$

which associates with each member  $x$  of  $A$  its singleton  $\{x\}$  is an injection. To complete the proof, we assume, toward a contradiction, that there exists a one to one correspondence

$$g : A \mapsto P(A)$$

which establishes that  $A \sim P(A)$  and we define the set

$$(4) \quad B = \{x \in A \mid x \notin g(x)\}.$$

Now  $B^1$  is a subset of  $A$  and  $g$  is a surjection, so *there must exist some  $b \in A$  such that  $B = g(b)$  (diagonalization)*, and (as for each  $x \in A$ ) either  $b \in B$  or  $b \notin B$ .

(\*) If  $b \in B$  then  $b \in g(b)$  since  $B = g(b)$ , so that  $b$  does not satisfy the condition which defines  $B$ , and hence  $b \notin B$ , contrary to hypothesis.

(\*\*) If  $b \notin B$ , then  $b \notin g(b)$ , so that  $b$  now satisfies the defining condition for  $B$  and hence  $b \in B$ , which again contradicts the hypothesis.

Thus we reach a contradiction from the assumption that the bijection  $g$  exists and the proof is complete.

## 1. THE RELATIVE COMPLEMENT

Let us read the above so-called Cantor's theorem and connect again to (4), i.e. the step of the definition of  $B$  within Cantor's argumentation. As previously observed, the relative complement can always be defined, thanks to (2). Accordingly let us define  $\overline{B} = A - B$  as the relative complement of  $B$  in  $A$ , i. e.

$$(5) \quad \overline{B} = \{x \in A \mid x \notin B\}.$$

One can easily see that

$$(6) \quad \overline{B} = \{x \in A \mid x \in g(x)\}.$$

In other words, being (5) legitimated by (2), whenever (4) is defined immediately (6) is defined too.

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<sup>1</sup>Notice that in  $\mathfrak{F}$  the definition of  $B$  is an example of (1), as one can easily see,  $B$  is defined within  $A$ .

Consequently we have in  $\mathfrak{ZF}$  the following situation

$$\begin{aligned} \mathbf{g} : A &\mapsto P(A) && \text{by assumption,} \\ B &= \{x \in A \mid x \notin \mathbf{g}(x)\} && \text{by Subset axiom,} \\ \overline{B} &= \{x \in A \mid x \in \mathbf{g}(x)\} && \text{by Subset axiom.} \end{aligned}$$

We can then state that  $B$  and  $\overline{B}$  are subsets of  $A$ . By its definition  $\mathbf{g}$  is a surjection and for each  $x \in A$  we have either  $x \in B$  or  $x \notin B$ , i.e. by (5) either  $x \in B$  or  $x \in \overline{B}$ . Let us reconsider the statement *there must exist some  $b \in A$  such that  $B = \mathbf{g}(b)$*  (*diagonalization*), within Cantor's argumentation. If such  $b$  exists, from  $B \neq \overline{B}$ , we obtain  $B = \mathbf{g}(b) \leftrightarrow \overline{B} \neq \mathbf{g}(b)$ , i.e.  $B = \mathbf{g}(b)$  or  $\overline{B} = \mathbf{g}(b)$  but not both. We have then the main consequence of taking into consideration the definition of the relative complement with respect to Cantor's argumentation in  $\mathfrak{ZF}$ . Applying (3) we obtain

$$(7) \quad (b \in \overline{B} \iff b \in \mathbf{g}(b)) \implies \overline{B} = \mathbf{g}(b),$$

hence by (6)

$$(8) \quad \overline{B} = \mathbf{g}(b).$$

Moreover since  $b \in B$  or  $b \in \overline{B}$  but not both, and  $B = \mathbf{g}(b)$  or  $\overline{B} = \mathbf{g}(b)$  but not both we have

$$(9) \quad B \neq \mathbf{g}(b).$$

Accordingly the assertion *there must exist some  $b \in A$  such that  $B = \mathbf{g}(b)$*  is false.

By the axiom schema of Subsets and the axiom of Extensionality, diagonalization can not be stated as true in  $\mathfrak{ZF}$ . Consequently (\*) and (\*\*) cannot be accomplished and Cantor's theorem does not hold in  $\mathfrak{ZF}$ . In fact we have only two cases

1.  $b \in B$  and  $\overline{B} = \mathbf{g}(b)$ , then  $b \notin \mathbf{g}(b)$  so that  $b$  satisfies the condition in (4) which defines  $B$ , and hence  $b \in B$ , accordingly to the hypothesis;
2.  $b \in \overline{B}$  and  $\overline{B} = \mathbf{g}(b)$ , then  $b \in \mathbf{g}(b)$  so that  $b$  satisfies condition in (6), and hence  $b \in \overline{B}$ , accordingly to the hypothesis.

We have thus established the following theorem.

**Theorem 1.** *By the definability of the relative complement, Cantor's proposition does not hold as a theorem in  $\mathfrak{ZF}$ .*

## 2. THE RESTRICTION ON UNIQUENESS

Let us leave aside now Cantor's argumentation. We assume to have a set  $A$  already defined in  $\mathfrak{ZF}$ . By the axiom schema of the Subsets we have

$$(I) \quad \vdash_{\mathfrak{ZF}} (b \in B \iff b \notin \mathbf{g}(b) \wedge b \in A),$$

which defines  $B$  as a subset of  $A$ . Since  $b \in A$  is true we obtain

$$(II) \quad \vdash_{\mathfrak{ZF}} (b \in B \iff b \notin \mathbf{g}(b)).$$

Furthermore by the axiom of Extensionality and the underlying laws for identity ( $\forall z(x \in z \iff y \in z) \iff x = y$ , [7] 25, 28)

$$(III) \quad \vdash_{\mathfrak{J}\mathfrak{F}} (b \in B \iff b \in \mathbf{g}(b)) \iff B = \mathbf{g}(b),$$

and therefore

$$(IV) \quad \vdash_{\mathfrak{J}\mathfrak{F}} (b \in B \iff b \notin \mathbf{g}(b)) \implies B \neq \mathbf{g}(b),$$

so that by (II) and (IV)

$$(V) \quad \vdash_{\mathfrak{J}\mathfrak{F}} B \neq \mathbf{g}(b).$$

Let us state in  $\mathfrak{J}\mathfrak{F}$

$$(VI) \quad B = \mathbf{g}(b),$$

then we attain

$$(VII) \quad \vdash_{\mathfrak{J}\mathfrak{F}} B = \mathbf{g}(b) \wedge B \neq \mathbf{g}(b),$$

accordingly  $\mathfrak{J}\mathfrak{F}$  turns out to be inconsistent. In simple terms, to state diagonalization,  $B = \mathbf{g}(b)$ , as true makes  $\mathfrak{J}\mathfrak{F}$  inconsistent. There is no need to yield diagonalization within the contest of a reasoning or argumentation. A definition like (I) leads to contradiction in any case. The explanation can be provided by the theory of definition which states the conditions and restrictions for defining proper equivalence in mathematics (see for example [11] 151-173). Definition (I) neglects a restriction embodied in the rules for proper definitions, established on the basis of the criterions of eliminability and non-creativity. Exactly as Russell's antinomy, definition of  $B$  drops the restriction on uniqueness, which is given when defining a new operation symbol (or a new individual constant, i.e. an operation symbol of rank zero) [4]. An equivalence like

$$O(x_1 \dots x_n) = y \iff \Phi$$

introducing a new operation symbol  $O$ , is a proper definition only if the formula

$$\exists! y \Phi$$

is derivable from the axioms and preceding definitions of the theory ([11] 158-159). In  $\mathfrak{J}\mathfrak{F}$  the uniqueness is ensured by the axiom of Extensionality (3), which implies that there exists *at most*, one set  $y$ , which contains exactly those elements  $x$  which fulfill the condition  $\varphi(x)$  in (1) ([7] 31). If  $B$  and  $\mathbf{g}(b)$  are two sets each of which contains exactly those elements  $b$  which fulfil the condition  $b \in \mathbf{g}(b)$ , then  $B$  and  $\mathbf{g}(b)$  are equal, see (III). Accordingly, there exists *at most*, one  $B$ , such that  $b \in \mathbf{g}(b)$ . Definition (I) implying  $B \neq \mathbf{g}(b)$ , (IV) and (V), neglects the restriction on uniqueness established by Extensionality and therefore the relative consistency embodied in the criterion of non-creativity ([11] 155; [4]). This explains why Extensionality blocks the derivation of the existence of some  $b \in A$  such that  $B = \mathbf{g}(b)$  in Cantor's argumentation. Moreover, this fulfils the criterion established by an editor, according to which, *to attack an argument, you must find something wrong in it*. We showed indeed that the definition of  $B$ , neglecting the restriction on uniqueness, is always wrong in  $\mathfrak{J}\mathfrak{F}$  and therefore a wrong *object sentence* in Cantor's argumentation [8].

When this results are regarded together with those presented in [3, 5] it arises clearly a similitude. If a set, or a predicate, is object of diagonalization then the

definition of its complement leads to the invalidity of the diagonalization itself. In Section 3 we shall apply this code of behavior to Cantor's diagonal argument.

### 3. CANTOR'S DIAGONAL ARGUMENT

In 1891 Cantor presented a striking argument which has come to know as Cantor's diagonal argument [2]. It runs as follows.

Consider the two elements  $m$  and  $w$ . Let  $M$  be the set whose elements  $E$  are sequences  $\langle x_1, x_2, \dots, x_v, \dots \rangle$  where each of  $x_1, x_2, \dots, x_v, \dots$  is either  $m$  or  $w$ .

**Cantor's proposition** *If  $E_1, E_2, \dots, E_v, \dots$  is any simply infinite sequence of elements of the set  $M$ , then there is always an element  $E_0$  of  $M$  which corresponds to no  $E_v$ .*

To prove this proposition, Cantor arranged a denumerable list of elements in an array.

$$\begin{array}{l} E_1 = \langle a_{1,1}, a_{1,2} \dots, a_{1,v}, \dots \rangle \\ E_2 = \langle a_{2,1}, a_{2,2} \dots, a_{2,v}, \dots \rangle \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ E = \langle a_{\mu,1}, a_{\mu,2} \dots, a_{\mu,v}, \dots \rangle \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

Each  $a_{\mu,v}$  is either  $m$  or  $w$ . Cantor defined a sequence  $b_1, b_2, b_3, \dots$ , where each element is  $m$  or  $w$ , and, if  $a_{v,v} = m$  then  $b_v = w$ , and if  $a_{v,v} = w$  then  $b_v = m$ . Let  $E_0 = \langle b_1, b_2, b_3, \dots \rangle$ . Then no  $E_v$  corresponds to  $E_0$ , by reason that  $b_v \neq a_{v,v}$ .

$E_1, E_2, \dots, E_v, \dots$  is any simply infinite sequence of elements of the set  $M$ , so that we can think to a definite infinite sequence of elements of  $M$  as follows.

$$\begin{array}{l} E_1^* = \langle a_{1,1}^*, a_{1,2}^* \dots, a_{1,v}^*, \dots \rangle \\ E_2^* = \langle a_{2,1}^*, a_{2,2}^* \dots, a_{2,v}^*, \dots \rangle \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ E^* = \langle a_{\mu,1}^*, a_{\mu,2}^* \dots, a_{\mu,v}^*, \dots \rangle \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

Each  $a_{\mu,v}^*$  is either  $m$  or  $w$  and if  $a_{\mu,v} = m$  then  $a_{\mu,v}^* = w$ , if  $a_{\mu,v} = w$  then  $a_{\mu,v}^* = m$ . Then  $b_v = a_{v,v}^*$  and  $E_0$  is never different on the  $v$ -th coordinate, so that it could even be for some  $E_v^*$  that

$$E_v^* = E_0.$$

Since  $E_1^*, E_2^*, \dots, E^* \dots$  is a simply infinite sequence of elements of  $M$ , previous Cantor's proposition is false.

In both the cases of Cantor's power set theorem and Cantor's diagonal argument, the definition of the complement of the object of diagonalization leads to the rejection of the diagonalization itself.

Working on logical complementation, Section 4, gives proof of the Axiom of Choice in  $\mathfrak{ZF}$ , and its refutation in a framework which is no longer  $\mathfrak{ZF}$ , on the basis of the universal validity of the first order logical truths.

#### 4. THE AXIOM OF CHOICE

Let us consider the following first order logic formula

$$(10) \quad \forall z \forall y \forall x [(x \in y \iff x \subseteq z) \iff (x \notin y \iff \neg(x \subseteq z))].$$

One can easily see it is a logically valid formula. We can then assume (10) holds in  $\mathfrak{ZF}$  (Zermelo Fraenkel set theory, [7]). From a comparison with the classical Axiom of Power Set

$$(11) \quad \forall z \exists y \forall x (x \in y \iff x \subseteq z)$$

where  $y$  is  $P(z)$ , the power-set of  $z$ , it follows immediately that (10) holds for the power-set because of its logical validity. We can then assume that (10) establishes the definition of  $P(z)$  as inseparable from the definition of its complement. We could think of  $\overline{P}(z)$  as defined by

$$(12) \quad \forall z \exists y \forall x (x \in y \iff \neg(x \subseteq z)).$$

Let us recall the Axiom of Choice as defined in ([7] 39-40, 53-55), to which the reader is referred for details.

(AC) *If  $t$  is a disjointed set which does not contain the null-set, its outer product  $\mathcal{O}Pt$  is different from the null-set.*

In other words, among the subsets of  $\bigcup t$  there is at least one whose intersection with each member of  $t$  is a singleton.

$\mathcal{O}Pt$  exists only if  $P(\bigcup t)$ , the set of the subsets of the union of  $t$ , exists. Immediately, by (10), AC is true, since, if  $t$  does not contain the null-set,  $P(\bigcup t)$  is never disjointed, and therefore there are selection sets of  $t$  and  $\mathcal{O}Pt$  is different from the null-set.

The reasons lie in the logical structure of (10), which states  $P(z)$ , namely  $y$ , to be a set excluding those parts  $x$  for which  $\neg(x \subseteq z)$  holds (see the component  $(x \notin y \iff \neg(x \subseteq z))$  in (10)).

Let us consider (10), with  $\emptyset \notin z$ .  $\neg(x \subseteq z)$  is true if  $(z \subset x) \vee (x \cap z = \emptyset)$ , hence  $P(z)$  is constructed in such a way that  $x$  and  $z$  are never disjointed (since  $\neg(z \subset x) \wedge \neg(x \cap z = \emptyset)$  holds for  $P(z)$  by (10)).

Now, consider  $t$  in AC.  $\emptyset \notin t$ , hence all the members of  $P(\bigcup t)$  are never disjointed, also if  $t$  is disjointed. Consequently  $\mathcal{O}Pt$  is different from the null-set.

The opposite holds for  $\overline{P}$ , because the members of  $\overline{P}(\bigcup t)$  are always disjointed, and even if  $t$  does not contain the null-set, its outer product is equal to the empty set. To visualize how  $\overline{P}$  gives rise to this situation, we can consider the following logically valid formula

$$(13) \quad \forall z \forall y \forall x [(x \in y \iff \neg(x \subseteq z)) \iff (x \notin y \iff \neg\neg(x \subseteq z))].$$

Since  $\emptyset \notin z$ , we can think of  $y$  as excluding those parts  $x$  such that  $x \subseteq z$  (and  $\neg((z \subset x) \vee (x \cap z = \emptyset))$ ). Thus for the parts  $x$  of  $y$  it holds always  $(x \cap z = \emptyset)$ .

To summarize we have proved that

$$[(1) \text{ and } P(z)] \implies \text{AC} \qquad [(4) \text{ and } \overline{P}(z)] \implies \text{not AC}.$$

We can then assert the following theorems.

**Theorem 2.** *The Axiom of Choice holds by (10).*

**Theorem 3.** *The negation of the Axiom of Choice holds by (13) (if  $\overline{P}$  is defined).*

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